# MATH 122B: HOMEWORK 3

**Suggested due date: August 22nd 2016**

(1) Compute the Laurent expansion of

$$\frac{e^z}{z(z^2+1)}, \quad 0 < |z| < 1.$$

(2) what are the singularities of the function $f(z) = \dfrac{z^4(z-1)}{\sin^2(\pi z)}$?

(3) Assume that $z = a$ is a pole of order $N$ of the function $f$. Show that it is a pole of order $N+1$ of the function $f'$.

(4) Compute the residue of the function

$$\frac{nz^{n-1}}{z^n - 1}$$

at its poles, including infinity. Then prove

$$\frac{nz^{n-1}}{z^n - 1} = \sum_{k=0}^{n-1} \frac{1}{z - \alpha_k}$$

where $\alpha_0, \ldots, \alpha_{n-1}$ are the roots of unity of order $n$.

(5) Let $C$ be a simple closed contour and $f$ holomorphic in and on $C$, with the possibly a finite number of poles inside $C$. Show that

$$\frac{1}{2\pi i} \int_C \frac{f'}{f} = Z - P$$

where $Z$ denotes the number of zeroes of $f$ and $P$ is the number of poles, both counting multiplicity, inside $C$.

(6) Compute $\displaystyle\int_{|z|=2} \frac{dz}{(z^{1000}+1)(z-3)}$.

(7) Compute the Fresnel integral $\displaystyle\int_0^\infty \sin(x^2)dx$.

(8) Compute $\displaystyle\int_0^\infty \frac{dx}{x^4+1}$.

(9) Compute $\displaystyle\int_{|z-1|=4} z^3 e^{1/z} dz$.

(10) Compute $\displaystyle\int_0^\infty \frac{\sin^2(x)}{x^2} dx$.

## Solutions

(1) Computing each Laurent series,

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$
$$\frac{1}{1+z^2} = \sum_{k=0}^{\infty} (-1)^k z^{2k}$$

We combine them to obtain

$$\frac{e^z}{z(z^2+1)} = \sum_{n=0}^{\infty} \left( \sum_{j=0}^{n} \frac{(-1)^j}{(n-2j)!} \right) z^{n-1}.$$

(2) The zeroes of $\sin(\pi z)$ are $n \in \mathbb{Z}$. Since the numerator has $z^4$, $z = 0$ is a removable singularity. Since $(z-1)$ is in the numerator but only of linear order, it only cancels with one order for $z = 1$, hence $z = 1$ is a simple pole, and all other positive and negative integers are poles of order 2.

(3) Since $f$ has a pole of order $N$ at $a$, in some punctured neighborhood,

$$f(z) = \frac{g(z)}{(z-a)^N}$$

where $g(a) \neq 0$. Then

$$f'(z) = \frac{(z-a)g'(z) - Ng(z)}{(z-a)^{N+1}} = \frac{h(z)}{(z-a)^{N+1}}$$

where $h(a) \neq 0$.

(4) The simple poles the $n$-th roots of unity $\{e^{2k\pi/n}\}_{k=0}^{n-1}$. The residue is

$$\mathrm{Res}(f, \alpha_k) = \frac{n(\alpha_k)^{n-1}}{n(\alpha_k)^{n-1}} = 1$$

The residue at infinity can be computed by

$$\begin{aligned} \mathrm{Res}(f, \infty) &= -\mathrm{Res}(\frac{1}{w^2} f(\frac{1}{w}), 0) \\ &= -\mathrm{Res}(\frac{n}{w(1-w^n)}, 0) = -n \end{aligned}$$

To prove the identity, we know that there is a partial fraction decomposition

$$\frac{nz^{n-1}}{z^n - 1} = \sum_{k=0}^{n-1} \frac{A_k}{z - \alpha^k}$$

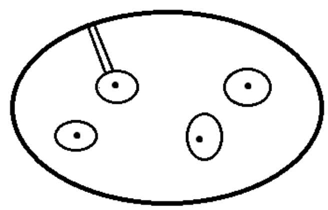
Solving for the coefficients, we have

$$nz^{n-1} = \sum_{k=0}^{n-1} A_k \Pi_{i \neq k}(z - \alpha_i)$$

Inserting $z = \alpha_k$, we have

$$A_k = \frac{n(\alpha_k)^{n-1}}{\Pi_{i \neq k}(\alpha_k - \alpha_i)} = \text{Res}(f, \alpha_k) = 1$$

(5) Let $z_1, \ldots, z_n$ be the zeroes and $p_1, \ldots p_m$ be the poles. Since the zeroes and poles are isolated, we can cover each by a ball so that the integral becomes the sum of the integral around each zero and pole



Let $C_i$ contain the zeroes and let $D_i$ contain the poles so

$$\frac{1}{2\pi i} \int_C \frac{f'}{f} = \frac{1}{2\pi i} \sum_i \left( \int_{C_i} \frac{f'}{f} \right) + \frac{1}{2\pi i} \sum_j \left( \int_{D_i} \frac{f'}{f} \right)$$

Around each zero, $z_i$, we can write $f$ as

$$f(z) = g(z)(z - z_i)^{n_i}$$

where $g(z_i) \neq 0$. Then

$$\frac{f'}{f} = \frac{g'}{g} + \frac{n_i}{(z - z_i)}$$

therefore

$$\frac{1}{2\pi i} \sum_i \left( \int_{C_i} \frac{f'}{f} \right) = \sum_i n_i = Z.$$

Around each pole, we can write $f$ as

$$f(z) = \frac{h(z)}{(z - p_i)^{m_i}}$$

where $h(p_i) \neq 0$. Hence

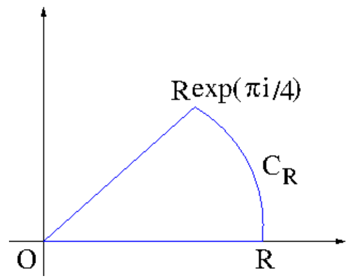
$$\frac{f'}{f} = \frac{h'}{h} - \frac{m_i}{(z - p_i)}$$

therefore

$$\frac{1}{2\pi i} \sum_i \left( \int_{D_i} \frac{f'}{f} \right) = -\sum_i m_i = -P.$$

(6) Use $\sum_{k=1}^{1000} \text{Res}(f, \alpha_k) = -\text{Res}(f, 3) - \text{Res}(f, \infty)$.

(7) Using the contour




with $\Gamma_R$ as the circular portion and $\gamma$ as the line towards the origin at $\pi/4$, we split the integral as

$$0 = \int_{C_R} e^{iz^2} dz = \int_0^R e^{ix^2} dx + \int_{\Gamma_R} e^{iz^2} dz + \int_\gamma e^{iz^2} dz.$$

By the inequality $-\sin(2\theta) \leq \frac{4\theta}{\pi}$ on $0 \leq \theta \leq \frac{\pi}{4}$, we have

$$\begin{aligned}\left|\int_R e^{iz^2} dz\right| &= \left|Ri \int_0^{\pi/4} e^{iR^2 e^{i2\theta}} e^{i\theta} d\theta\right| \\ &\leq R \int_0^{\pi/4} e^{-R^2 \sin(2\theta)} d\theta \\ &\leq R \int_0^{\pi/4} e^{-4R^2\theta/\pi} d\theta \\ &= -\frac{R\pi}{4R^2}(e^{-R^2} - 1) \to 0\end{aligned}$$

as $R \to \infty$. To compute the integral along $\gamma$, we have

$$\begin{aligned}\int_\gamma e^{iz^2} dz &= -e^{i\pi/4} \int_0^R e^{-r^2} dr \\ &\to -\frac{\pi}{2}(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}),\end{aligned}$$

as $R \to 0$. Matching the real and imaginary parts, we get

$$\int_0^\infty \sin(x^2) dx = \frac{\sqrt{2\pi}}{4}$$

(8) Similar and easier method as above, answer is $\frac{\pi}{2\sqrt{2}}$.

(9) Straightforward application of residue theorem, computing the Laurent series, we have

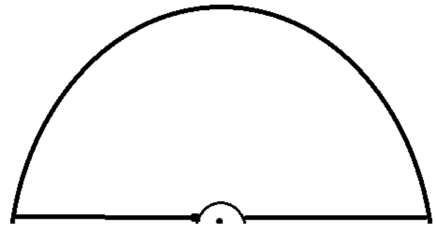
$$z^3 e^{1/z} = z^3(1 + \frac{1}{z} + \frac{1}{2z^2} + \frac{1}{3!z^3} + \frac{1}{4!}z^4 + O(\frac{1}{z^5}))$$

so the $1/z$ term is $\frac{1}{4!}$. Thus by residue theorem, the answer is $\frac{2\pi i}{4!}$.

(10) Consider the integral

$$4\int_0^\infty \frac{\sin^2(x)}{x^2} dx = \int_{-\infty}^\infty \frac{1 - \cos(2x)}{x^2} dx.$$

To integrate, we integrate along the contour $C_R$



where the outer circle is $\Gamma_R$ and the inner circle is $\gamma_r$. Then the integral splits as

$$\int_{C_R} \frac{1-e^{i2z}}{z^2}dz = \int_{-R}^{-r} \frac{1-e^{i2z}}{z^2}dz + \int_{r}^{R} \frac{1-e^{i2z}}{z^2}dz + \int_{\Gamma_R} \frac{1-e^{i2z}}{z^2}dz + \int_{\gamma_r} \frac{1-e^{i2z}}{z^2}dz$$

We can show that $\int_{\Gamma_R} \frac{e^{1-i2z}}{z^2}dz \to 0$ as $R \to \infty$. So we integrate the smaller circle by parametrizing $z = re^{i\theta}$ for $\theta$ from $\pi$ to 0, so that

$$\begin{aligned}
\int_{\gamma_r} \frac{1-e^{i2z}}{z^2}dz &= -\int_0^{\pi} \frac{1-e^{i2re^{i\theta}}}{r^2e^{2i\theta}} rie^{i\theta}d\theta \\
&= -i\int_0^{\pi} \frac{-(2ire^{i\theta}) - \sum_{n=2}^{\infty} \frac{(i2re^{i\theta})^n}{n!}}{r} e^{-i\theta}d\theta \\
&= -2\pi + O(r) \to -2\pi
\end{aligned}$$

as $r \to 0$. Hence the answer is $\pi/2$.